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# ON SIMPLE GROUPS AND SIMPLE SINGULARITIES

BY

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ABSTRACT

We give a proof of a characteristic $p$ version of Brieskorn's theorem, namely, that if $G$ is a simply connected simple algebraic group of type $A$, $D$ or $E$ over an algebraically closed field $k$ whose characteristic is very good for $G$, then the categorical quotient morphism $G \to G//G_{ad}$ yields, when restricted to a general slice through a point $P$ in the subregular unipotent orbit in $G$, a miniversal deformation of the rational double point over $k$ of the same type as $G$.

## 1. Introduction

Suppose that $G$ is a split simply connected simple Chevalley group of rank $r$ and type $A, D$ or $E$ over $\mathbb{Z}$ with adjoint group $G_{ad}$ and that $k$ is an algebraically closed field whose characteristic $p$ is very good for $G$. Grothendieck conjectured that the categorical quotient morphism $G \to G//G_{ad}$ yields, when restricted to a general slice through a point $P$ in the subregular unipotent orbit in $G \otimes k$, a miniversal deformation (also known as a semi-universal deformation) of the rational double point (RDP) over $k$ of the same type as $G$. Brieskorn [B] proved this, in the context of Lie algebras over $k$ rather than groups over $\mathbb{Z}$, provided that $p = 0$. A complete proof was published by Slodowy [Sl], who extended the result to the case where $p > 4\,\mathrm{Cox}(G) - 2$. His proof uses the Jacobson–Morozov lemma to construct a slice that is equivariant under a certain $\mathbb{G}_m$-action, and then exploits the fact (which is verified case by case) that the degrees of the co-ordinates on a miniversal deformation space are the degrees of the Weyl group.

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This note, which is a sequel to Hinich's papers [H1,2], presents a proof of Brieskorn's theorem, in the context of groups, when $p$ is very good. It avoids the coincidence just mentioned, but depends upon those listed as (1)–(3) below. We show first that Grothendieck's construction yields a miniversal deformation of the minimal resolution $\widetilde{Y}$ of $Y$. We then deduce the result for $Y$ via a very easy local Torelli theorem for $\widetilde{Y}$ in characteristic zero. (The idea of analyzing the deformations of an RDP by first analyzing those of its minimal resolution appears in [P].) In particular, we recover the existence of a Weyl group action on $\mathrm{Def}_{\widetilde{Y}}$ whose quotient is $\mathrm{Def}_Y$ (where $\mathrm{Def}_Z$ denotes a miniversal deformation space of an object $Z$, when it exists) and that a smoothing of an RDP has a simultaneous resolution, without base-change, if and only if there is no monodromy on the vanishing cohomology.

Hinich works in the context of Lie algebras. In [H1], he proves, by Grothendieck duality and Grauert–Riemenschneider vanishing, that the normalization of any nilpotent orbit closure has rational Gorenstein singularities, while in [H2] he gives a quick identification when $p = 0$ of the singularities along the subregular nilpotent orbit. This argument relies upon (1)–(2):

(1) In any characteristic, and even in mixed characteristic, a surface singularity is an RDP if it has a resolution with trivial canonical class, and this resolution is then minimal [Li]. (In fact, Hinich gives a direct proof that the exceptional curves have self-intersection $-2$.) This follows from general results about rational surface singularities.

(2) In good characteristic, RDPs are determined up to isomorphism by the combinatorial structure of a minimal resolution. This is something of a lucky coincidence, and is only known to be true from an inspection of Artin's list [A2].

If also $p$ is very good, then the the orbit maps of $G_{ad}$ are separable, and Hinich's argument extends without material change to identify the singularities.

The other fact that we exploit, that again depends upon an inspection of Artin's list, is:

(3) In very good characteristic, the dimension of a miniversal deformation space equals the rank (the number of curves in the minimal resolution).

Hinich's result identifies the singularities also when $G$ is not simply laced, and Slodowy's results cover this case also; they describe the induced deformation as a particular invariant subspace of a miniversal deformation of the singularity. We have not tried to determine whether the approach taken here can be made to work in this context.

*Notation:* Vector bundles will be denoted by capital italic letters, and their sheaves of sections by the corresponding script letters. A miniversal deformation space of a space $V$ will be denoted by $\mathrm{Def}_V$, when it exists. We say that $p$ is **good** if $p \neq 2, 3, 5$ for type $E_8$, $p \neq 2, 3$ for type $E_6$ and $E_7$, and $p \neq 2$ for type $D$. Moreover, $p$ is **very good** if also $p$ does not divide $r + 1$ for type $A_r$.

## 2. Grothendieck's construction

Fix, over $\mathbb{Z}$, a maximal torus $T$ of $G$, with Weyl group $W$, and then fix a Borel subgroup $B$ containing $T$. Denote by $e$ the identity of $T \otimes k$ and, by abuse of notation, its image in $T/W$. Let $\phi$: $G \to G//G_{ad}$ be the categorical quotient by the adjoint action. If $\chi_i$ is the trace of the fundamental representation $\rho_i$ of $G$, which is defined over $\mathbb{Z}$, then there are natural ring homomorphisms $\mathbb{Z}[\chi_1, \ldots, \chi_r] \to \mathcal{O}_G^{G_{ad}} \to \mathcal{O}_T^W$ whose composite is, by exponential invariant theory, an isomorphism. One can show [St] that the other maps are also isomorphisms, so that $G//G_{ad}$ is identified with $\operatorname{Spec} \mathbb{Z}[\chi_1, \ldots, \chi_r]$ and the natural map $T \to G//G_{ad}$ induces an isomorphism $T/W \to G//G_{ad}$. The unipotent variety $N$ is $\phi^{-1}(e)$. It is flat over $\mathbb{Z}$ and all geometric fibres are normal.

For any parabolic subgroup $P$ of $G$ and $V$ a connected subgroup of $G$ that is normalized by $P$ (all over $\mathbb{Z}$), there is a proper collapsing map $(G \times V)/P \to G$, where $P$ acts on $G \times V$ by $p(g, v) = (gp^{-1}, p(v))$, induced by $G \times V \to G$ : $(g, v) \mapsto g(v)$.

We can identify $(G \times P)/P$ with the incidence variety $\{(P', v) | v \in P'\} \subset (G/P) \times G$, where $G/P$ is identified with the variety of conjugates of $P$, and then the collapsing map is identified with the projection to $G$.

Put $\widetilde{X} = (G \times B)/B$ and let $\widetilde{X} \to X \to G$ be the Stein factorization of the collapsing map. Let $U$ denote the unipotent radical of $B$, so that the map $T \to B/U$ is an isomorphism. Via this, $B$ acts trivially on $T$ and there is a projection $\psi$: $\widetilde{X} \to T$. Then $\widetilde{X}_e = \psi^{-1}(e)$ is the subvariety $((G \times U)/B) \otimes k$.

THEOREM 2.1: (1) (Springer) *The collapsing map $\widetilde{X} \to G$ induces $\widetilde{X}_e \to N$, and this is a desingularization.*
(2) (Grothendieck) *There is a commutative diagram*

$$\begin{array}{ccccc} \widetilde{X} & \xrightarrow{\rho} & X & \longrightarrow & G \\ \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\phi_1} & & \downarrow{\scriptstyle\phi} \\ T & \xrightarrow{=} & T & \xrightarrow{\pi} & G//G \end{array}$$

*where the right square is Cartesian over some neighbourhood of $\phi(e)$ and $\psi$ is smooth.*

*Proof:* (1) See [St, pp. 129–130].
(2) See [Sl, p. 50]. ∎

The next point is to describe a part of the exceptional locus of $\widetilde{X} \to X$. Fix a simple root $\beta$ and let $P_\beta \supset B$ be the corresponding rank 1 parabolic subgroup, so that $P_\beta/B \cong \mathbb{P}^1$. Let $V_\beta$ denote the soluble radical of $P_\beta$ and $U_\beta$ the unipotent radical. Then there is a commutative exact diagram

$$\begin{array}{ccccccccc} 1 & \longrightarrow & U_\beta & \longrightarrow & V_\beta & \longrightarrow & \beta^\perp & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & \longrightarrow & U & \longrightarrow & B & \longrightarrow & T & \longrightarrow & 1, \end{array}$$

where the vertical maps are inclusions and $\beta^\perp$ is the kernel of the character $\beta$. Note that $V_\beta$ has codimension 2 in $B$, so that $E_\beta = (G \times V_\beta)/B$ has codimension 2 in $\widetilde{X}$, and that the image of the composite $E_\beta \to \widetilde{X} \to T$ is $\beta^\perp$.

Lemma 2.2: *The exceptional locus of $\widetilde{X} \to X$ contains $E_\beta$.*

*Proof:* The collapsing map $E_\beta \to G$ factors through the $\mathbb{P}^1$-bundle $\pi_\beta\colon E_\beta \to (G \times V_\beta)/P_\beta$. ∎

Proposition 2.3: *Suppose that $C$ is a fibre of $\pi_\beta\colon E_\beta \to (G \times V_\beta)/P_\beta$. Then*
*(2.3.1) $N_{E_\beta/\widetilde{X}}|_C \cong O(-1)^2$ and*
*(2.3.2) every deformation of $C$ in $\widetilde{X}$ lies in $E_\beta$.*

*Proof:* (2.3.1) From the description of $E_\beta \hookrightarrow \widetilde{X}$ as an inclusion of associated bundles over $G/B$, it follows that $N_{E_\beta/\widetilde{X}} \cong (G \times V_\beta \times (\operatorname{Lie} B/\operatorname{Lie} V_\beta))/B$. We identify $C$ with the subvariety $(P_\beta \times \{1\})/B$ of $E_\beta$; then $N_{E_\beta/\widetilde{X}}|_C \cong (P_\beta \times (\operatorname{Lie} B/\operatorname{Lie} V_\beta))/B$. Choose a copy $G_1$ of $\mathrm{SL}_2$ in a Levi subgroup of $P_\beta$ so that $B_1 = B \cap G_1$ is a Borel subgroup of $G_1$. So $N_{E_\beta/\widetilde{X}}|_C \cong (G_1 \times \operatorname{Lie} B_1)/B_1$. From the exact sequence

$$0 \to \operatorname{Lie} U_1 \to \operatorname{Lie} B_1 \to \operatorname{Lie} T_1 \to 0$$

of $B_1$-modules, where $U_1$ is the unipotent radical of $B_1$ and $T_1$ a torus in $B_1$, it follows that $N_{E_\beta/\widetilde{X}}|_C$ is either $O \oplus O(-2)$ or $O(-1)^2$ over $\mathbb{P}^1$. So if $H^0(\mathbb{P}^1, N_{E_\beta/\widetilde{X}}|_C) \neq 0$, then $\operatorname{Lie} B_1$ contains a $B_1$-invariant vector. But it does not, and so $N_{E_\beta/\widetilde{X}}|_C \cong O(-1)^2$.

(2.3.2) It is enough to show that $\mathrm{Hilb}_{\widetilde{X}}$ is smooth at $[C]$ and that $\dim_{[C]}\mathrm{Hilb}_{\widetilde{X}} = \dim(G\times V_\beta)/P_\beta$. For this, it is enough to show that $\mathcal{N}_{C/\widetilde{X}} \cong \mathcal{O}(-1)^2\oplus\mathcal{O}^{\dim G-3}$. There is an exact sequence

$$0\to N_{C/E_\beta}\to N_{C/\widetilde{X}}\to N_{E_\beta/\widetilde{X}}|_C\to 0,$$

so that now (2) follows from (1) and the fact that $N_{C/E_\beta}$ is free, since $C$ is a fibre of $E_\beta\to(G\times V_\beta)/P_\beta$. ∎

Now suppose that $p$ is very good. Then the action of $G_{ad}$ on any orbit is smooth and it follows [Sl, pp. 60–69] that there is an $(r+2)$-dimensional complete slice of $G$ through a geometric point $P$ of the subregular unipotent orbit $N_1$ whose inverse image is smooth. Localize the lower row by completing at $e$. In particular, this makes a base change from $\mathbb{Z}$ to the Witt vectors $\mathbb{W}$, which will be the base ring henceforth. Denote the slices on the upper row by a superscript dagger $^\dagger$, except that we write $X_e^\dagger = Y$ and $\widetilde{X}_e^\dagger=\widetilde{Y}$.

Define $F_\beta = ((G\times U_\beta)/B)\otimes k$. Note that $F_\beta = E_\beta\cap\widetilde{X}_e$ and $F_\beta$ is a divisor in $\widetilde{X}_e$. Define $C_\beta = F_\beta\cap\widetilde{Y} = E_\beta\cap\widetilde{Y}$.

PROPOSITION 2.4: [H2] (2.4.1) *The exceptional locus of* $\widetilde{Y}\to Y$ *is* $\bigcup_\beta C_\beta$.
(2.4.2) *Each* $C_\beta$ *is an irreducible* $(-2)$*-curve and the configuration that they form is the same as the Dynkin diagram of* $G$.
(2.4.3) $(Y,P)$ *is an RDP of the same type as* $G$ *and* $\widetilde{Y}\to Y$ *is its minimal resolution.*

*Proof:* As already mentioned, Hinich [H2] gives a transparent proof of this, depending only on the facts (1)–(2) listed in the introduction. He proves first that the dualizing sheaf $\omega_{\widetilde{Y}}$ is trivial, which ensures that $(Y,P)$ is an RDP and $\widetilde{Y}\to Y$ is its minimal resolution. He then identifies the exceptional locus, which is enough. He works in the context of complex Lie algebras but, except for replacing $H^2(G/B,\mathbb{Z})$ by $\mathrm{Pic}(G/B)$ and $T^*(G/B)$ by $(G\times U)/B$, which is easily seen to have trivial dualizing sheaf, his proof carries over unchanged.

## 3. The miniversal deformation of $\widetilde{Y}$

Suppose that $f\colon \widetilde{Y}\to(Y,P)$ is the minimal resolution of an RDP and that $E_1,\dots,E_r$ are the exceptional curves in $\widetilde{Y}$. Any deformation of $\widetilde{Y}$ can be blown down to a deformation of $Y$, giving a morphism $\mathrm{Def}_{\widetilde{Y}}\to\mathrm{Def}_Y$.

PROPOSITION 3.1: *Assume that* $\operatorname{char} k = p$ *is very good for the type of* $Y$.
(3.1.1) $\mathrm{Def}_{\widetilde{Y}}$ *is smooth over* $\operatorname{Spec}\mathbb{W}$ *of relative dimension* $r$.

(3.1.2) *The spaces* $H_i \subset \mathrm{Def}_{\widetilde{Y}}$ *of deformations where* $E_i$ *is preserved are smooth transverse divisors over* Spec $\mathbb{W}$.

*Proof:* There is an exact sequence

$$0 \to T_{\widetilde{Y}}(-\log \sum E_i) \to T_{\widetilde{Y}} \to \oplus \mathcal{N}_{E_i/\widetilde{Y}} \to 0.$$

Since $H^2(\widetilde{Y}, T_{\widetilde{Y}}) = H^2(\widetilde{Y}, T_{\widetilde{Y}}(-\log \sum E_i)) = 0$, by reason of dimension, $\mathrm{Def}_{\widetilde{Y}}$ is smooth of dimension at least $r$ over $\mathbb{W}$. Suppose that $\widetilde{\mathfrak{Y}} \to \mathrm{Def}_{\widetilde{Y}}$ is a miniversal deformation of $\widetilde{Y}$. Let $R$ be the quasi-separated algebraic space that represents [A1] the functor $\mathrm{Res}_{\widetilde{\mathfrak{Y}} \to \mathrm{Def}_{\widetilde{Y}}}$. Then [A1, Theorem 3] there is a henselization $\widetilde{R}$ of $R$ that surjects finitely to $\mathrm{Def}_Y$, while also [A1, Lemma 3.3] there is a smooth morphism $R \to \mathrm{Def}_{\widetilde{Y}}$. Now an inspection of Artin's lists [A2] shows that $\mathrm{Def}_Y$ is of dimension $r$ over Spec $\mathbb{W}$, and (3.1.1) follows.

Finally, the tangent space of $H_i$ is $\ker(H^1(\widetilde{Y}, T_{\widetilde{Y}}) \to H^1(E_i, \mathcal{N}_{E_i/\widetilde{Y}}))$, and we are done. ∎

Lemma 3.2: *Suppose that* $Y$ *is of type* $A_1$ *and that* $f\colon \widetilde{\mathcal{Y}} \to S = \mathrm{Spec}\, \mathbb{W}\{t\}$ *is a deformation of* $\widetilde{Y}$. *Put* $S_0 = \mathrm{Spec}\, \mathbb{W}\{t\}/(p)$ *and* $\widetilde{\mathcal{Y}}_0 = \widetilde{\mathcal{Y}} \times_S S_0$. *Then* $f$ *embeds into* $\mathrm{Def}_{\widetilde{Y}}$ *if* $\mathcal{N}_{E_1/\widetilde{\mathcal{Y}}_0} \cong \mathcal{O}(-1) \oplus \mathcal{O}(-1)$.

The converse is also true, but we shall not need it.

*Proof:* Suppose that $S_1$ is the miniversal deformation space of $\widetilde{Y}$ and that $\alpha\colon S \to S_1$ is the morphism induced by $f$. Since $S$ and $S_1$ are $\mathbb{W}$-flat, $\alpha$ is an embedding if and only if $\alpha \otimes 1_k$ is so. Hence it is enough to show that $f_0$ embeds into the miniversal deformation.

Put $\Sigma = \mathrm{Spec}\, \mathcal{O}_{S_0}/(t^2)$. If $f_0$ does not embed into the miniversal deformation, then $\widetilde{\mathcal{Y}}_0 \times \Sigma \to \Sigma$ is trivial, so that $\mathcal{N}_{E_1/\widetilde{\mathcal{Y}}_0} \cong \mathcal{N}_{E_1/\widetilde{Y}} \oplus \mathcal{O}$. ∎

Proposition 3.3: *The map* $E_\beta^\dagger \to X^\dagger$ *is a* $\mathbb{P}^1$*-bundle over its image.*

*Proof:* Let $\tau\colon B \to T$ be the projection. Put $\widetilde{X}_\beta = (G \times \tau^{-1}(\beta^\perp))/B = \psi^{-1}(\beta^\perp)$. Put $X_\beta = \phi_1^{-1}(\beta^\perp)$; then it is enough to show that the fibres of $E_\beta^\dagger \to X_\beta^\dagger$ are copies of $\mathbb{P}^1$.

Since $U_\beta$ is the transverse intersection of $V_\beta$ and $U$ inside $\tau^{-1}(\beta^\perp)$, it follows that $E_\beta$ and $\widetilde{X}_e$ meet transversely inside $\widetilde{X}_\beta$. So $E_\beta \cap \widetilde{X}_e$, and so $E_\beta^\dagger \cap \widetilde{X}_e^\dagger$, is smooth. By 2.3.2 (rather, its proof) $E_\beta^\dagger \cap \widetilde{X}_e^\dagger$ is a single copy of $\mathbb{P}^1$. ∎

THEOREM 3.4: *$\widetilde{X}^\dagger \to T$ is a miniversal deformation of $\widetilde{Y}$.*

*Proof:* As in the proof of 3.2, it is enough to pass from $\mathbb{W}$ to $k$ and work over $k$. We do this, but suppress the subscript $k$ in what follows.

The map $\widetilde{X}^\dagger \to T$ is induced from a morphism $g\colon T \to \mathrm{Def}_{\widetilde{Y}}$. By 2.5.2, $E_\beta^\dagger \cap \widetilde{Y}$ is a copy $C_\beta$ of $\mathbb{P}^1$. Let $H_\beta \subset \mathrm{Def}_{\widetilde{Y}}$ be the hyperplane consisting of deformations where $C_\beta$ is preserved. Then $g^{-1}(H_\beta)$ is, by 2.3.2, set-theoretically equal to the image in $T$ of $E_\beta^\dagger$, which is just $\beta^\perp$.

Choose co-ordinates $\{t_\beta\}$ on $\mathrm{Def}_{\widetilde{Y}}$ and $\{z_\beta\}$ on $T$ so that $H_\beta = (t_\beta)_0$ and $\beta^\perp = (z_\beta)_0$. Then $g$ is identified with a homomorphism $k\{t_{\beta_1}, \ldots, t_{\beta_r}\} \to k\{z_{\beta_1}, \ldots, z_{\beta_r}\}$ such that $t_{\beta_i} \mapsto u_i z_{\beta_i}^{n_i}$ for some $n_i \in \mathbb{N}$ and some unit $u_i$.

Suppose that $n_i > 1$. Write $\beta_i = \beta$. Take a slice $\Delta_1 = \mathrm{Spec}\, k\{t_1\}$ of $\mathrm{Def}_{\widetilde{Y}}$ through a general point $P$ of $H_\beta$. Put $\Delta = \Delta_1 \times_{\mathrm{Def}_{\widetilde{Y}}} T$; then $\Delta \cong \mathrm{Spec}\, k\{t\}$ and $\Delta \to \Delta_1$ is of degree $n_i$. Denote fibre products with $\Delta$ by a subscript.

By 3.3, the fibre of $E_\beta^\dagger$ over $P$ is a fibre $C$ of $\pi_\beta$. Then $\widetilde{X}_\Delta^\dagger \to \Delta$ induces, by restricting to an étale neighbourhood of $C$, a deformation of the minimal resolution of an $A_1$-singularity which, by construction, does not embed into the miniversal deformation. So $N_{C/\widetilde{X}_\Delta^\dagger} \cong \mathcal{O} \oplus \mathcal{O}(-2)$. Moreover, $C$ is a fibre of $E_\beta \to Z_\beta$, so that $N_{C/\widetilde{X}_\Delta^\dagger} \cong N_{E_\beta/\widetilde{X}}|_C$. This contradicts 2.3.1.

So $n_i = 1$ and $g$ is an isomorphism. ∎

## 4. The miniversal deformation of $Y$

We shall show that $G^\dagger \to S = G//G_{ad}$ is a miniversal deformation of $Y = X_e^\dagger$. Let $\mathfrak{Y} \to V$ be a miniversal deformation of $Y$, so that there is a Cartesian square

$$\begin{array}{ccc} G^\dagger & \longrightarrow & \mathfrak{Y} \\ \big\downarrow & & \big\downarrow \\ S & \xrightarrow{\ \gamma\ } & V. \end{array}$$

LEMMA 4.1: *$\gamma$ is finite.*

*Proof:* If not, then there is a curve $\Delta$ through $e$ in $S$ such that the induced family $G_\Delta^\dagger \to \Delta$ is a trivial deformation of $Y$. Then there is a curve $\Delta_1$ in $T$, dominating $\Delta$, such that the induced family $\widetilde{X}_{\Delta_1}^\dagger \to \Delta_1$ is a trivial deformation of $\widetilde{Y}$, contradicting Theorem 3.4. ∎

First, assume that $p = 0$, so that we can take $k = \mathbb{C}$.

For any parameter space $B$, denote the discriminant by $D_B$ and put $B \smallsetminus D_B = B_0$.

Fix a base point $s_0 \in S_0$, with image $v_0 \in V_0$. We get an identification $G^{\dagger}_{s_0} = \mathfrak{Y}_{v_0}$. Let $M$ denote the lattice $H^2_c(\mathfrak{Y}_{v_0}, \mathbb{Z})$, which we regard as a subgroup of $H^2(\mathfrak{Y}_{v_0}, \mathbb{Z})$. Since there is a simultaneous resolution $\widetilde{X}^{\dagger} \to T$, we can and do fix an identification of $M$ with the lattice $L$ generated by the exceptional curves in $\widetilde{Y}$. $M$ also contains the lattice $N$ generated by the vanishing cycles.

Since there is a simultaneous resolution over $T$, there is no monodromy over any open subset of $T$ on $M$. Let $U_0 \to V_0$ be the étale Galois covering that kills the monodromy on $M$ and $U \to V$ the normalization of $V$ in the function field of $U_0$; then $T \to V$ factors through $U$, so that there is a commutative square

$$\begin{array}{ccc} T & \longrightarrow & S \\ {\scriptstyle\delta}\downarrow & & \downarrow{\scriptstyle\gamma} \\ U & \longrightarrow & V. \end{array}$$

Let $T_0$ denote the inverse image of $V_0$ in $T$.

LEMMA 4.2: *The Hodge–de Rham spectral sequence yields an isomorphism* $\alpha \colon H^2(\widetilde{Y}, \mathbb{C}) \to H^1(\widetilde{Y}, \Omega^1)$.

*Proof:* Since $E_1^{pq} = 0$ for $q > 1$, the map $\alpha$ exists and is surjective. Since both sides have dimension $\operatorname{rank}(G)$, the result follows. ∎

Consider the family $\psi \colon \widetilde{X}^{\dagger} \to T$ and construct a period map $P : T \to H^1(\widetilde{Y}, \Omega^1)$ as follows. Pick a generator $\omega$ of $\omega_{G^{\dagger}/S}$; this lifts to a generator of $\Omega^2_{\widetilde{X}^{\dagger}/T}$. Then $\omega|_{\widetilde{X}^{\dagger}_t}$ can be transported along a path from $t$ to $e$, and so defines a class in $H^2(\widetilde{Y}, \mathbb{C})$. Define $P(t) = P_\omega(t) = \alpha([\omega|_{\widetilde{X}^{\dagger}_t}])$.

PROPOSITION 4.3: *$P$ is an isomorphism at $e$.*

*Proof:* The derivative of $P$ at $e$ is a map $H^1(\widetilde{Y}, T_{\widetilde{Y}}) \to H^1(\widetilde{Y}, \Omega^1)$. As is well known [K, 3.4, especially 3.4.12], this is identified with contraction against $\omega|_{\widetilde{Y}}$. Since $\omega|_{\widetilde{Y}}$ induces an isomorphism $T_{\widetilde{Y}} \to \Omega^1_{\widetilde{Y}}$, it does so on $H^1$. ∎

COROLLARY 4.4: *The map $\delta \colon T \to U$ is an isomorphism.*

*Proof:* Since there is no monodromy over $U_0$, $P|_{T_0}$ factors through $U_0$. Now the result follows from 4.3. ∎

THEOREM 4.5: *$\gamma: S \to V$ is an isomorphism.*

*Proof:* Put $\Gamma = \mathrm{Gal}(U/V)$. Then $\Gamma$ is generated by reflexions in the vanishing cycles [Lo]. Since $N \subset M$ the vanishing cycles are roots in $M$, so that $\Gamma \subset W$. However, $\#W$ divides $\#\Gamma$, by 4.4, so that $S \to V$ is a finite map of smooth germs and is of degree 1. ∎

Now drop the assumption that $p = 0$.

THEOREM 4.6: *$\gamma$ and $\delta$ are isomorphisms over* $\mathrm{Spec}\,\mathbb{W}$.

*Proof:* Put $K = \mathrm{Frac}(\mathbb{W})$, so that $\gamma$, $\delta$ are finite morphisms of smooth germs over $\mathbb{W}$ and $\gamma \otimes 1_{\bar{K}}$, $\delta \otimes 1_{\bar{K}}$ are isomorphisms. Then $\gamma$, $\delta$ are isomorphisms. ∎

COROLLARY 4.7: *Suppose that $\mathfrak{Y} \to D$ is a deformation of an RDP $Y$ in very good characteristic and that $D$ is normal. Then there is a simultaneous resolution $\widetilde{\mathfrak{Y}} \to D$ if and only if the monodromy action on the vanishing cohomology is trivial.*

*Proof:* We identify $T$ with $\mathrm{Def}_{\widetilde{Y}}$ and $S$ with $\mathrm{Def}_Y$. So there is a simultaneous resolution $\widetilde{\mathfrak{Y}} \to D$ if and only if any (or all) of the natural morphisms $D \to \mathrm{Def}_Y \cong S$ factors through $T$. Since $\delta$ is an isomorphism, we are done. ∎

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